Review: Taylor Polynomials - 11/9/16

1 Theory

Goal: Find a degree n polynomial that approximates a function $f(x)$ at a point a.

Definition 1.0.1 *n* ***factorial*** *is $n! = n \cdot (n-1) \cdots 2 \cdot 1$.*

Example 1.0.2 $3! = 3 \cdot 2 \cdot 1 = 6$, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

Definition 1.0.3 *The* ***nth order Taylor polynomial of*** *f* ***at*** *a is*

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Notice that first order Taylor polynomials are just the linear approximations.

2 Examples

Example 2.0.4 *Find T_4 of $f(x) = \ln(x)$ at $x = 1$. How many derivatives do we need? Four, since we're trying to find T_4.*

$$\begin{aligned} f'(x) &= \frac{1}{x} \\ f''(x) &= -\frac{1}{x^2} \\ f'''(x) &= \frac{2}{x^3} \\ f^{(4)}(x) &= -\frac{6}{x^4} \end{aligned}$$

Then following what we found above,

$$T_4(x) = 0 + \frac{1}{1}(x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \frac{6}{4!}(x-1)^4.$$

Example 2.0.5 *Find T_3 of $f(x) = e^x$ at $x = 0$. How many derivatives do we need? Three, since we're trying to find T_3. Luckily, the derivatives are all e^x, so $f^{(n)}(0) = e^0 = 1$. Then*

$$T_3(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3.$$

Example 2.0.6 *Find T_n of $f(x) = e^x$ at $x = 0$. Then*

$$T_n(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n.$$

Example 2.0.7 *Find T_5 of $f(x) = \cos(x)$ at $x = \frac{\pi}{2}$. How many derivatives do we need? Five, since we're trying to find T_5.*

$$\begin{aligned} f'(x) &= -\sin(x) \\ f''(x) &= -\cos(x) \\ f'''(x) &= \sin(x) \\ f^{(4)}(x) &= \cos(x) \\ f^{(5)}(x) &= -\sin(x) \end{aligned}$$

Since $\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$, then we have

$$T_5(x) = -\frac{1}{1!}\left(x - \frac{\pi}{2}\right) + \frac{1}{3!}\left(x - \frac{\pi}{2}\right)^3 - \frac{1}{5!}\left(x - \frac{\pi}{2}\right)^5.$$

Example 2.0.8 *Find T_5 of $f(x) = \cos(x)$ at $x = 0$. We still use the same derivatives, but now $\cos(0) = 1$ and $\sin(0) = 0$. Then*

$$T_5(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4.$$

Example 2.0.9 *Find T_3 of $f(x) = 5x^3 + 7x - 5$ at $x = 0$. We need to find three derivatives:*

$$\begin{aligned} f'(x) &= 15x^2 + 7 \\ f''(x) &= 30x \\ f'''(x) &= 30 \end{aligned}$$

Then

$$T_3(x) = -5 + \frac{7}{1!}x + \frac{0}{2!}x^2 + \frac{30}{3!}x^3 = -5 + 7x + 5x^3.$$

This is the same as our original polynomial! This is what we hoped would happen, since the Taylor polynomial is supposed to be a polynomial approximation of $f(x)$.

Practice Problems Look at the practice problems that we did in class. The problems and solutions are posted on the website.